Approximating Cayley diagrams versus Cayley graphs

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Abstract. We construct a sequence of finite graphs that weakly converge to a Cayley graph, but there is no labelling of the edges that would converge to the corresponding Cayley diagram. A similar construction is used to give graph sequences that converge to the same limit, and such that a Hamiltonian cycle in one of them has a limit that is not approximable by any subgraph of the other. We give an example where this holds, but convergence is meant in a stronger sense. This is related to whether having a Hamiltonian cycle is a testable graph property.

By a *diagram* we will mean a graph with edges oriented and labelled by elements of some given set (of "colors"). By a *Cayley diagram* we mean a Cayley graph when we do not forget that edges are oriented and labelled by elements of the generating set. For simplicity, when an edge is oriented both ways with the same label (that is, when the labelling generator has degree 2), we will represent it by an unoriented edge in the Cayley diagram, and sometimes refer to it as a 2-cycle. A *rooted graph (diagram)* is a graph (diagram) with a distinguised vertex called the root; a rooted isomorphism between rooted graphs G and H is an isomorphism that maps root into root. A rooted labelled-isomorphism between rooted diagrams G and H is a rooted isomorphism that preserves orientations and labels of the edges.

Let $\mathcal{G}$ be the set of rooted isomorphism classes of countable connected rooted graphs. Let $\tilde{\mathcal{G}}$ be the set of rooted isomorphism classes of connected rooted diagrams. We can introduce a metric on $\mathcal{G}$ by saying that the distace between $G, H \in \mathcal{G}$ is 2^{-r} if r is the largest integer such that the r-neighborhood of the root in G is rooted isomorphic to the r-neighborhood of the root in H. We can define distance on $\tilde{\mathcal{G}}$ similarly. It is easy to check that the generated topology makes $\mathcal{G}$ ($\tilde{\mathcal{G}}$) into a complete separable metric space.

Suppose that G_n is a sequence of finite graphs. Then we can define a probability measure μ_n on $\mathcal{G}$ by picking a vertex o uniformly at random as the root, and projecting the resulting measure to $\mathcal{G}$. Now, say that G_n *converges* to a probability measure μ on $\mathcal{G}$, if the μ_n weakly converges to μ (i.e., for each bounded continuous function $f : \mathcal{G} \to \mathbb{R}$ we have $\int_{\mathcal{G}} f d\mu_n \to \int_{\mathcal{G}} f d\mu$). This convergence is often called Benjamini-Schramm convergence; see [AS] or [AL] for more details. If G is some transitive graph, then we can define a Dirac delta measure μ on $\mathcal{G}$ that is supported on the rooted isomorphism class of (G, o), where

o is an arbitrary point. If G_n converges to this μ, then we will say that G_n converges to G, or that G is approximated by G_n. Similarly, if G is quasi-transitive, there is a natural finitely supported probability measure on $\{(G, o_1), \ldots, (G, o_m)\}$, where $\{o_1, \ldots, o_m\}$ in G is a traversal for the orbits of the automorphism group of G. The same definitions and terminology apply for diagrams instead of graphs (where "rooted isomorphism" is replaced by "rooted labelled-isomorphism").

Less formally, convergence of G_n to a transitive G means that for any r, the proportion of vertices x in G_n whose r-neighborhood with x as a root is rooted isomorphic to the r-neighborhood of o in G tends to 1 as $n \to \infty$. It is a central open question whether any *unimodular* transitive graph can be approximated by a sequence of finite graphs. See [AL] for the definition of unimodularity (which is a necessary condition for the existence of such an approximation), and for more details on what we have introduced. Cayley graphs are unimodular, and a finitely generated group is called *sofic*, if it has a finitely generated Cayley diagram that is approximable by a sequence of finite diagrams. (There are several equivalent definitions of soficity, see [ESz] or [P] for history and references.) The interest in whether every group is sofic comes partly from the fact that many conjectures are known to hold for sofic groups. A nice brief survey on the subject is [P].

By definition, if a sequence $\tilde{G}_n$ of finite diagrams converges to a Cayley diagram $\tilde{G}$, then the underlying graphs G_n converge to the underlying Cayley graph G. It is natural to ask, whether the converse is true, or whether the approximability of a Cayley graph by finite graphs implies that the group is sofic. The next two questions phrase this in increasing difficulty. The second one seems to have been asked by several people independently. The first one was proposed by Russell Lyons at a workshop in Banff [ASz].

QUESTION 1. Suppose that a sequence G_n of finite graphs converges to a Cayley graph G, and let $\tilde{G}$ be a Cayley diagram with underlying graph G. Is there a sequence of diagrams $\tilde{G}_n$ such that if we forget about orientations and labels of edges in $\tilde{G}_n$ we get G_n, and such that the sequence $\tilde{G}_n$ converges to the diagram $\tilde{G}$?

QUESTION 2. Suppose that a Cayley graph of a group Γ is approximable by a sequence of finite graphs. Is Γ then sofic?

We give a negative answer to Question 1 in Theorem 3. This indicates that the existence of an approximating sequence for a Cayley graph may not help directly in the construction for an approximation of the Cayley diagram. In fact, it is reasonable to think that to answer Question 2 might be as difficult as the question whether every group is sofic. The difficulty of Question 2 is further illustrated by the fact, as explained to us by Gábor Elek, that some Burger-Mozes groups are known to have a Cayley graph that is the

direct product of two regular trees, even though these groups are simple and not known to be sofic. The product of trees is clearly approximable by a sequence of finite graphs, hence a positive answer to Question 2 would imply that these groups are sofic. See IV.9. in [dlH] for more on isometric Cayley graphs and Burger-Mozes groups.

If a sequence of graphs (diagrams) G_n weakly converges to a graph (diagram) G, we will write $G_n \to G$. Given a graph or diagram G and vertex $v \in V(G)$, we denote the r-neighborhood of v in G by $B_G(v, r)$.

THEOREM 3. *There exists a Cayley diagram $\tilde{G}$ such that the corresponding Cayley graph G is the weak limit of a sequence G_n of randomly rooted finite graphs, but there is no sequence of diagrams $\tilde{G}_n$ that would weakly converge to $\tilde{G}$ and such that the graph underlying $\tilde{G}_n$ is G_n.*

Proof. Consider $G = T \times \mathbf{C_4}$, where T is the 3-regular tree, C_4 is the cycle of length 4, and in the direct product two edges are adjacent by definition iff they are equal in one coordinate, and adjacent in the other. In other words, we have four copies T_1, T_2, T_3, T_4 of the 3-regular tree (that we will also call *fibers*), some isomorphisms $\phi_1 : T_1 \to T_2$, $\phi_2 : T_2 \to T_3$, $\phi_3 : T_3 \to T_4$, $\phi_4 : T_4 \to T_1$ such that $\phi_4^{-1} = \phi_3 \circ \phi_2 \circ \phi_1$; and G consists of $T_1 \cup T_2 \cup T_3 \cup T_4 \cup K$, where K denotes the set of all edges of the form $\{v, \phi_i(v)\}$ (in particular, K consists of cycles of length 4).

Let $\tilde{G}$ be the following diagram. We consider the Cayley diagram $\tilde{T}$ of $\mathbb{Z}*\mathbb{Z}_2 = \langle a, b|b^2\rangle$. Make T_i a Cayley diagram labelled-isomorphic to $\tilde{T}$, and do it in such a way that the ϕ_i are labelled-isomorphisms. To define labels on elements of K, we will use colors c and d. Namely, for each 4-cycle in K, color the edges by c and d alternatingly. Do it in such a way that if the edge between v and $\phi_i(v)$ has label c, then for all neighbors w of v in T_i, the edge $\{w, \phi_i(w)\}$ will have color d; and similarly with c and d interchanged.

We claim that the resulting $\tilde{G}$ is a Cayley diagram. Consider

$$\langle a, b, c, d|b^2, c^2, d^2, cdcd, ada^{-1}c, aca^{-1}d, bcbd\rangle.$$

To see that the corresponding Cayley diagram is indeed the diagram $\tilde{G}$ that we defined, note that the latter has a cycle space generated by 2- and 4-cycles. The relators given here, together with some of their conjugates of reduced lengths 4, are exactly the words read along 2- and 4-cycles on a given vertex.

Now, let H_n be a sequence of 3-regular graphs with girth tending to infinity and independence ratio less than $1/2 - \epsilon < 1/2$. See [B] for the construction of such a sequence (with $\epsilon = 1/26$). Define $G_n = H_n \times C_4$. Clearly, $G_n \to G$. Suppose now, that there is a diagram $\tilde{G}_n$ with underlying graph G_n such that $\tilde{G}_n$ weakly converges to $\tilde{G}$. Let H_n^1, H_n^2,

H_n^3, H_n^4 be the four copies of $\tilde{H}_n$ in $\tilde{G}_n$ (we will call them *fibers* of $\tilde{G}_n$). Fix a point o of G in T_1. Say that $x \in \tilde{G}_n$ is *R-good*, if there exists a rooted labelled-isomorphism from $B_{\tilde{G}_n}(x, R)$ to $B_{\tilde{G}}(o, R)$.

For $R \geq 4$, the ball $B_{\tilde{G}}(o, R)$ has only one rooted labelled-isomorphism to itself, the identity. This is so because every rooted labelled-isomorphism has to preserve edges in $T_1 \cup T_2 \cup T_3 \cup T_4$, and the only two rooted labelled-isomorphisms that respect the labels and orientations on T_1, T_2, T_3, T_4 can be the identity and one that switches T_2 and T_4. The latter, however, switches edges of labels c and d, hence it is not a rooted labelled-isomorphism. As a consequence of the fact just proved, if a graph is rooted labelled-isomorphic to $B_{\tilde{G}}(o, R)$ ($R \geq 4$), then there is a unique isomorphism between them. For each R-good x, each i and each ι rooted labelled-isomorphism from $B_{\tilde{G}_n}(x, R)$ to $B_{\tilde{G}}(o, R)$, every vertex of $B_{G_n}(x, R) \cap H_n^i$ is mapped into the same T_j by ι (that is, if two point are in the same fiber, then they are mapped into the same fiber by the rooted labelled-isomorphism). This is so because preserving labels on the edges means in particular that edges within a fiber (of label a or b) are mapped into edges within a fiber (the ones having label a or b). There is at most one such rooted labelled-isomorphism (since if there were more, that would give a nontrivial rooted labelled-isomorphism from $B_{\tilde{G}}(o, R)$ to itself, as observed above). We have obtained that for every R-good $x \in \tilde{G}_n$ ($R \geq 4$), there is a unique rooted labelled-isomorphism ι_x from $B_{\tilde{G}_n}(x, R)$ to $B_{\tilde{G}}(o, R)$, and it maps fibers into fibers (in a bijective way).

Since ι_x preserves fibers and is an isomorphism, it either changes the cyclic order of $H_n^1, H_n^2, H_n^3, H_n^4$ (meaning $\iota_x(H_n^1 \cap B_{\tilde{G}_n}(x, R)) = B_G(o, R) \cap T_j, \iota_x(H_n^2 \cap B_{\tilde{G}_n}(x, R)) = B_G(o, R) \cap T_{j-1}, \ldots$), or preserves the cyclic orientation. Let $\overrightarrow{S}_n$ be the set of R-good points x in $\tilde{G}_n$ where ι_x preserves the cyclic order, and $\overleftarrow{S}_n$ be the set of those where it reverses the cyclic order. We claim that if x and y are R-good and adjacent in $\tilde{G}_n$, then ι_x and ι_y give different orientations. To see this, let the c-edges adjacent to x and y in $\tilde{G}_n$ be $\{x, x'\}$ and $\{y, y'\}$ respectively. Then $\iota_x(x')$ and $\iota_x(y')$ are in different fibers, hence x' and y' are in different fibers too. On the other hand $\iota_x(x')$ and $\iota_y(y')$ are in the same fiber by definition, hence one of ι_x and ι_y has to preserve orientation and the other one has to reverse it.

We conclude that $\overrightarrow{S}_n$ is an independent set, and also $\overleftarrow{S}_n$ is an independent set. By the choice of the H_n we then have $|\overrightarrow{S}_n \cap H_n^i|/|H_n^i| \leq \frac{1}{2} - \epsilon$ for every i, and similarly for the $\overleftarrow{S}_n$. Hence

$$|\overrightarrow{S}_n \cup \overleftarrow{S}_n|/|\tilde{G}_n| = \sum_{i=1}^{4} |\overrightarrow{S}_n \cap H_n^i|/4|H_n| + |\overleftarrow{S}_n \cap H_n^i|/4|H_n| \leq 1 - 2\epsilon.$$

This is uniform in n, contradicting the fact that the proportion of R-good points in $\tilde{G}_n$ (that is, $\overrightarrow{S}_n \cup \overleftarrow{S}_n$) tends to 1. ∎

Gábor Elek has asked the following question. A positive answer would show that having a Hamiltonian cycle is a testable graph property. (A property being testable is, vaguely, the following. Given a finite graph G, can we decide by sampling a bounded number of balls in it, whether there is a graph G' with the property in question, and such that one can transform G into G' by changing an at most ϵ proportion of the edges in G? See [L] for the precise definition) . A result of this type is the one in [EL], where it is shown that for a convergent graph sequence the matching ratio (that is, the ratio of the size of a maximal matching and the size of the graph) also has a limit. This implies that the matching ratio is a testable graph parameter, [E]. See [L] for the relevance of parameter testing and its connection to graph sequences.

QUESTION 4. Let G_n and H_n be two graph sequences, converging to the same (random) G. Suppose that G_n contains a Hamiltonian cycle C_n (whose limit is then a biinfinite path). Is there a subgraph in H_n whose limit is the same?

We construct an example where convergence to the same limit fails only in a stronger sense, namely, that there is no subgraph D_n in H_n such that the pair (H_n, D_n) would converge to the same pair, as (G_n, C_n). (For C_n subgraph of G_n on the same vertex set, one can think about the pair (G_n, C_n) as a diagram on G_n, simply by coloring edges of C_n to one color and edges outside of C_n to another one. Convergence of the pairs (G_n, C_n) can then be defined as convergence of the respective diagrams.) Our example will use the one given in Theorem 3.

THEOREM 5. *There are two sequences, G_n and K_n, that converge to the same Cayley graph G, and such that K_n contains a Hamiltonian cycle C_n such that the pair (K_n, C_n) converges to (G, F), where $F \subset G$ is a unimodular random graph, but G_n does not have any subgraph D_n such that (G_n, D_n) would converge to (G, F).*

Proof. Consider $G = T \times C_4$ as in Theorem 3.

Let $G_n = H_n \times C_4$ be as in Theorem 3. We have seen that the limit of G_n is G.

The other sequence, K_n will also be the direct product of a graph B_n and C_4, and it will have the property that it contains a Hamiltonian cycle such that every other edge of the Hamiltonian cycle is an edge "coming from C_4 (by which we mean an edge of the form $\{(x, v), (x, w)\}$, $x \in B_n$, $v, w \in C_4$ adjacent). So, consider a bipartite graph B_n, with the following properties. It is 3-regular, it contains a Hamiltonian cycle, it has "upper set" $U_n = U$ and "lower set" $L_n = L$ both containing $2n + 1$ vertices, and

the girth tends to infinity as $n \to \infty$. We will construct K_n as follows. First, define a bipartite *directed* graph K'_n on vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = \{x^1_v : v \in U\}$, $V_3 = \{x^3_v : v \in U\}$, $V_2 = \{x^2_w : w \in L\}$, $V_4 = \{x^4_w : w \in L\}$, and set of directed edges $\{(x^i_s, x^{i+1}_t) : \{s,t\} \in E(B_n)\}$, where $i+1$ is modulo 4 (and similarly later for such indices, without further mention). That is, for each pair V_i, V_{i+1}, we "copy" B_n on $V_i \cup V_{i+1}$, (V_i playing the role of U iff i is odd), and orient the edges from V_i towards V_{i+1}. In particular, K'_n has $2(4n+2)$ vertices, each having indegree 3 and outdegree 3, and all edges going out of V_i go to V_{i+1}. To finish, let $K_n = H$ be a bipartite graph of $4(4n+2)$ vertices, whose vertex set is obtained by doubling every vertex w of K'_n to get the twins $\bar{w}$, $\hat{w}$. Let $\bar{w}$ and $\hat{v}$ be adjacent in K_n iff there is a (directed) edge from w to v in K'_n. Further, connect each pair of twins $\bar{w}$, $\hat{w}$ by an edge, and call the edges of this type *blue* edges. Finally, if $x^1_v \in V_1$, $x^3_v \in V_3$ (with a $v \in V(U_n)$), then connect $\bar{x}^1_v$ and $\hat{x}^3_v$ by an edge, and connect $\hat{x}^1_v$ and $\bar{x}^3_v$ by an edge. Similarly, if $x^2_v \in V_2$, $x^4_v \in V_4$ (with a $v \in V(L_n)$), then connect $\bar{x}^2_v$ and $\hat{x}^4_v$ by an edge, and connect $\hat{x}^2_v$ and $\bar{x}^4_v$ by an edge. Call the edges of these type *yellow* edges. Observe that the colored edges of K_n form cycles of lengths 4, each colored by yellow and blue alternatingly. More precisely, note that K_n is isomorphic to $B_n \times C_4$. To see this, note that each of the four sets $\{\bar{x}^i_v : v \in U_i\} \cup \{\hat{x}^{i+1}_w : w \in L_{i+1}\}$, $i = 1, 3$, and $\{\bar{x}^i_v : v \in L_i\} \cup \{\hat{x}^{i+1}_w : w \in U_{i+1}\}$, $i = 2, 4$ induces a graph isomorphic to B_n. We will refer to these four sets as fibers. Colored edges of K_n then correspond to edges coming from C_4 in the direct product. In particular, it is clear that K_n converges to G. Consider now the set S of blue edges with one endpoint in a fiber C^1 and the other endpoint in fiber C^2. In the direct product, the 4-cycles that correspond to neighboring vertices have alternating colorings, hence the endpoints of S in C^1 form an independent set (since B_n is bipartite). We will refer to this as the "independence property".

We claim that K_n contains a Hamiltonian cycle. To see this, let the vertices in a Hamiltonian cycle of B_n be $v_1, v_2, \ldots, v_{4n+2}$, listed in their order along the cycle. The respective vertices $x^1_{v_1}, x^2_{v_2}, x^3_{v_3}, v^4_{v_4}, x^1_{v_5}, x^2_{v_6}, \ldots, x^2_{v_{4n+2}}, x^3_{v_1}, x^4_{v_2}, x^1_{v_3}, x^2_{v_4}, \ldots, x^4_{v_{4n+2}}$ determine a Hamiltonian directed cycle in K'_n. These edges can be projected into K_n, and if we add the (blue) edge between each pair $\bar{x}^i_v$, $\hat{x}^i_v$, we get a Hamiltonian cycle C_n of K_n. Every second edge on C_n is blue.

Now, let Ω be the set of edges of G not in the fibers (that is, edges coming from C_4). We have seen that local isomorphisms from K_n to G map colored edges to edges in Ω. Hence the limit of C_n in G is a biinfinite path F that has every other edge in Ω. Fibers are also preserved, thus by the "independence property" we obtain for the set of edges of $F \cap \Omega$ with one endpoint in a fiber C^1 and the other in a fiber C^2, that the set of their endpoints in C^1 is independent.

Suppose now that there is a subgraph $D_n \subset G_n$ such that (G_n, D_n) would converge to (G, F). We proceed similarly as in the proof of Theorem 3. Fix $o \in V(G)$, and let X be the set of R-good points x such that the (unique) local isomorphism from $B_{G_n}(x, R)$ to $B_G(o, R)$ does not change the (previously fixed) orientation of the fibers. By the same argument as in the last two paragraphs of the proof of Theorem 3, X is an independent set. Furthermore, its density is larger than $(1-\epsilon)/2$ if n is large enough, since $G_n \to G$. This contradicts the assumption on the size of the largest independence set in G_n.

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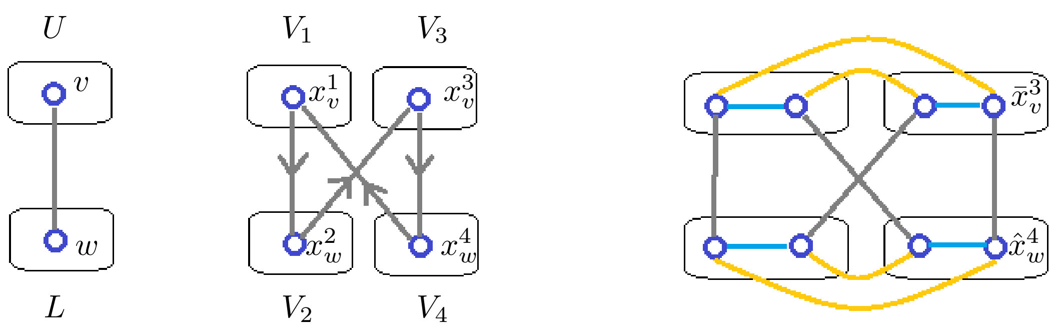

The scheme of constructing K_n' and K_n from B_n.

Acknowledgements. I am very grateful to Russell Lyons and Gábor Elek for motivating questions and helpful discussions. I also thank Gábor Lippner for his comments on the manuscript.

Research supported by Sinergia grant CRSI22-130435 of the Swiss National Science foundation.

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